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Observations Concerning the Magnitude of $t\bar{t}$ Threshold Effects on Electroweak Parameters

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Abstract

We discuss a recent analysis of $t\bar{t}$ threshold effects and its implications for the determination of electroweak parameters. We show that the new formulation, when applied to the ρ parameter for $m_t \approx 158$ GeV, gives a result of similar magnitude to those previously obtained. In fact, it is quite close to our "resonance" calculation. We also present a simple estimate of the size of the threshold effects based on an elementary Bohr-atom model of $t\bar{t}$ resonances.

1 Introduction

The analysis of threshold effects involving heavy quarks and their contribution to the determination of electroweak parameters has been the subject of a number of studies in the past [1,2,3,4,5]. Very recently, F.J. Ynduráin has discussed, in the framework of the Coulombic approximation, $t\bar{t}$ threshold effects on the renormalized vacuum-polarization function $\Pi(s) - \Pi(0)$ associated with conserved vector currents [6]. Here $\Pi(s)$ is the unrenormalized function defined according to $\Pi^V_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$, with $s = q^2$. For $m_t = \sqrt{3}m_Z \approx 158$ GeV and $s = m_Z^2$, the author of Ref. [6] finds that the threshold effects are significantly smaller than the perturbative $\mathcal{O}(\alpha_s)$ calculation. From this observation he concludes that threshold effects are generally small for $s \ll 4m_t^2$ and that the "large" results reported in Ref. [5] concerning their contribution to electroweak parameters are not supported by his "detailed, rigorous calculation." However, $t\bar{t}$ threshold effects influence electroweak parameters chiefly through the ρ parameter. The function $\Pi(s) - \Pi(0)$, although very important in its own right, has very little to do with the effects discussed in Ref. [5]. In fact, it is well known that heavy particles of mass $m^2 \gg s$ decouple in this amplitude. For this reason, it should be obvious that the leading effects discussed in

Ref. [5] do not arise from this amplitude. Thus, conclusions drawn on the work of Ref. [5] from the discussion of $\Pi(s) - \Pi(0)$ are without foundation. In order to show in the simplest possible way what the correct conclusions ought to be, in Section 2 we apply the formulation of Ref. [6] to the study of leading threshold contributions to the ρ parameter for $m_t \approx 158$ GeV and other values of m_t. In principle, this requires only minor modifications of the relevant formulae of Ref. [6], mainly in the prefactors. However, for reasons explained in that Section, we find it necessary to re-evaluate the threshold corrections of that paper. We then find that, contrary to the conclusions of Ref. [6], this leads to results similar in magnitude to those reported in Ref. [5]. In fact, the approach of Ref. [6] gives threshold corrections to the ρ parameter quite close to our own "resonance" calculation. In order to make more transparent the size of these leading threshold corrections relative to the perturbative $\mathcal{O}(\alpha_s)$ calculations, in Section 3 we present a simple estimate based on an elementary Bohr-atom model of toponia. We also briefly comment on the shifts induced in electroweak parameters and the magnitude of $\mathcal{O}(\alpha_s^2)$ corrections.

2 Threshold corrections to the ρ parameter

For large m_t, the dominant threshold effects discussed in Ref. [5] can be related to corrections to the ρ parameter. As the current $\bar{\psi}_t\gamma^\mu\psi_t$ is conserved and the ρ parameter is defined at $s = 0$, it is clear that $t\bar{t}$ threshold effects arise in this case from the contributions of the axial-vector current. As explained in Ref. [5], on account of the Ward identities such contributions involve $\Im m\, \lambda^A(s, m_t, m_t)$, where λ^A is a longitudinal part of the axial-vector polarization tensor $\Pi^A_{\mu\nu}$. Furthermore, for non-relativistic, spin-independent QCD potentials, $-\Im m\, \lambda^A$ can be identified to good approximation with $\Im m\, \Pi$ (in Ref. [5], Π is called $\Pi^V(s)/s$). In particular, we note that both amplitudes receive contributions from nS states. One then finds that, in the formulation Ref. [5], the leading threshold correction to the ρ parameter is given by (cf. Eqs. (5.1, 5.2a) of Ref. [5])

$$\delta(\Delta\rho)_{thr} = -\frac{G_F}{2\pi\sqrt{2}} \int ds'\; \Im m\, \Pi_{thr}(s'). \tag{1}$$

Here $\Im m\, \Pi_{thr}(s')$ denotes contributions from the threshold region not taken into account in the usual perturbative $\mathcal{O}(\alpha_s)$ calculation.

Our aim is to employ the analysis of Ref. [6] to calculate Eq. (1) and then to compare the answer with our results. In the formulation of that paper, based on the Coulombic approximation, there are two contributions to $\Im m\, \Pi_{thr}(s')$. One of them arises from the toponium resonances and is called $\Im m\, \Pi_{pole}(s')$. The other represents a summation of $(\alpha_s/v)^n$ $(n = 1, 2, \ldots)$ terms, integrated over a small range above threshold, after subtracting corresponding $\mathcal{O}(\alpha_s)$ contributions. The first one can be obtained from the expression [6]

$$\Im m\, \Pi_{pole}(s) = N_c \sum_n \delta(s - M_n^2) \frac{\left|\tilde{R}^{(0)}_{n0}(0)\right|^2}{M_n} \left[1 + \frac{3\beta_0\alpha_s}{2\pi}\left(\ln\frac{n\mu}{C_F\alpha_s m_t} + \psi(n+1) - 1\right)\right], \tag{2}$$

where $N_c = 3$, $C_F = (N_c^2-1)/(2N_c) = 4/3$, $n_f = 5$, $\beta_0 = 11 - 2n_f/3 = 23/3$, M_n is the mass of the nS toponium resonance in a Coulombic potential, $\left|\tilde{R}_{n0}^{(0)}(0)\right|^2 = C_F^3\tilde{\alpha}_s^3(\mu)m_t^3/(2n^3)$ is the square of its radial wave function at the origin, and $\tilde{\alpha}_s(\mu) = \alpha_s(\mu)(1+b\alpha_s(\mu)/\pi)$, with $b = \gamma_E(11N_c-2n_f)/6+(31N_c-10n_f)/36 \approx 3.407$ for toponia [7]. Inserting Eq. (2) into Eq. (1), we obtain

$$\begin{aligned}\delta(\Delta\rho)_{pole} &= -x_t\pi\zeta(3)C_F^3\tilde{\alpha}_s^3(\mu)\left[1+\frac{3\beta_0\alpha_s}{2\pi}\left(\ln\frac{\mu}{C_F\alpha_s m_t}-\gamma_E\right)\right.\\ &\quad\left.+\frac{3\beta_0\alpha_s}{2\pi\zeta(3)}\sum_{n=2}^{\infty}\frac{1}{n^3}\left(\ln n+\sum_{k=2}^{\infty}\frac{1}{k}\right)\right],\end{aligned}\tag{3}$$

where $x_t = (N_cG_Fm_t^2/8\pi^2\sqrt{2})$ is the Veltman correction to the ρ parameter [8]. In the numerical evaluation of the threshold corrections to $\Pi(s)-\Pi(0)$ reported in Ref. [6], μ is chosen to be m_Z, $m_t = \sqrt{3}m_Z$ is assumed, and $\alpha_s(m_Z) = 0.115 \pm 0.01$ is taken. Unless stated otherwise, we shall adopt these values in the following. Furthermore, $(15/16)\pi\zeta(3)C_F^3\tilde{\alpha}_s^3(\mu)$ is found to equal 1.81×10^{-2}, the summation $\sum_{n=2}^{\infty}$ is neglected, and $\{1+(3\beta_0\alpha_s/2\pi)[\ln(\mu/C_F\alpha_sm_t)-\gamma_E]\}$ is given as 1.08. It is apparent that the numerical value given in Ref. [6] for the last factor is too low. The correct value is 1.315. Inclusion of the neglected sum raises the expression between square brackets in Eq. (3) to 1.437, instead of 1.08 [9]. This leads to

$$\begin{aligned}\delta(\Delta\rho)_{pole} &= -\frac{16}{15}1.816\times10^{-2}\cdot1.437\,x_t\\ &= -0.0278\,x_t.\end{aligned}\tag{4}$$

The contribution from the small range above threshold can be gleaned from Ref. [6]. Using $v = (1-4m_t^2/s)^{1/2}$ as integration variable (v is the top-quark velocity in the center-of-mass frame) and approximating $(1-v^2)^{-2} \approx 1$ in the integrand, the contribution to $\Delta\rho$ equals $-(16/15)\delta_{thr}x_t$, where δ_{thr} is a quantity studied in Ref. [6]. For $s \ll 4m_t^2$, δ_{thr} can be written as

$$\delta_{thr} = \frac{15}{4}\int_0^{v_0}dv\,v\left[\frac{B(v)}{1-e^{-B(v)/v}}-v-\frac{\pi}{2}C_F\alpha_s(m_t)\right],\tag{5}$$

$$B(v) = \pi C_F\alpha_s(\mu)\left\{1+\frac{\alpha_s}{\pi}\left[b+\frac{\beta_0}{2}\left(\ln\frac{C_F\alpha_s\mu}{4m_tv^2}-1\right)\right]\right\},\tag{6}$$

where an overall factor $(1-v^2/3)$ has been omitted under the integral. In Eq. (5), the term involving $B(v)$ represents a summation of $(\pi C_F\alpha_s/v)^n$ contributions valid for large values of this parameter, while the last two correspond to the subtraction of the perturbative calculation up to $\mathcal{O}(\alpha_s)$ in the small-v limit. We have evaluated the latter at m_t, as this is demonstrably the proper scale to be employed in the perturbative calculation [5]. We note that the integrand of Eq. (5) is renormalization-group invariant through $\mathcal{O}(\alpha_s^2)$ [10]. In Ref. [6], the value $v_0 = \pi C_F\alpha_s(m_t)/\sqrt{2} \approx 0.314$ is chosen, the exponential and terms of higher order in α_s are neglected, and the answer given as

$$\delta_{thr} = \frac{15}{16}\pi^3C_F^3\alpha_s^2(m_t)\left(\frac{1}{2}\alpha_s(\mu)-\frac{\sqrt{2}}{3}\alpha_s(m_t)+\frac{b}{\pi}\alpha_s^2(\mu)+\frac{\beta_0}{2\pi}\alpha_s^2(\mu)\ln\frac{\mu}{2\pi^2C_F\alpha_sm_t}\right).\tag{7}$$

Although the analytic summation of $(\pi C_F \alpha_s/v)^n$ terms is theoretically interesting, there are unfortunately a number of problems in the evaluation of δ_{thr} carried out in Ref. [6]:

1. Evaluation of Eq. (7) as it stands gives a negative result, $\delta_{thr} = -3.82 \times 10^{-3}$ [11], which obviously contradicts the well-known fact that multi-gluon exchanges lead to an enhancement of the $t\bar{t}$ excitation curve.

2. Equation (7) is not renormalization-group invariant, as the coefficient of $\alpha_s(\mu)$ does not match correctly that of $\ln \mu$. Subject to the approximations explained before, the correct, renormalization-group invariant expression is obtained by including an additional term $[\alpha_s(\mu) - \alpha_s(m_t)]/2$ within the parentheses of Eq. (7). We note in passing that this additional term may be traced to a change of scale in the last term of Eq. (5), from the arbitrary value μ to the proper physical choice m_t. For $\mu = m_Z$, the value of the corrected expression is $\delta_{thr} = -3.83 \times 10^{-4}$, i.e., essentially zero, and very different from the value reported in Ref. [6].

3. As we have a near cancellation of relatively large contributions and, moreover, the result should be positive, it is clear that, for the chosen value of v_0, the neglect of the exponential in Eq. (5) is not justified. In fact, evaluating numerically Eq. (5) with the exponential included, we find $\delta_{thr} = 1.55 \times 10^{-2}$. This value is positive, as it should be, and much larger than the answer obtained without the exponential. Actually, it is 2.9 times larger than the result reported in Ref. [6].

We also note that the integrand of Eq. (5) vanishes at $1.002\, v_0$, where v_0 is defined above Eq. (7). Thus, v_0 is a reasonable value to use as the upper limit of integration because at that point the resummed series and the perturbative $\mathcal{O}(\alpha_s)$ contributions nearly coincide; evaluation of the integrals up to the point where the integrand actually vanishes leads to negligible changes. On the other hand, a possible weakness of the method is that v_0 is rather large and for such values of v it is not clear that the resummed expression is valid. Nevertheless, for our present purpose, which is the evaluation of the contribution to $\Delta\rho$ according to the prescriptions of Ref. [6], we use the above value of v_0. For the same reason, we use $\mu = m_Z$, although this is not a characteristic scale in connection with the ρ parameter; using the appropriate value, $\mu = m_t$, would make the radiative correction and the overall result slightly smaller. However, we include the effect of an additional overall factor $(1 - v^2/3)/(1 - v^2)^2$, which should be appended to the integrand of Eq. (5) in the case of $\Delta\rho$; it increases the result by about 4.3%. We then find that the contribution to $\Delta\rho$ from the range $0 \le v \le v_0$ above threshold is $-(16/15) 1.55 \times 10^{-2} \cdot 1.043\, x_t = -0.0172\, x_t$. Combining this result with Eq. (4), we find that, after correcting the errors discussed above, the formulation of Ref. [6] leads to $\delta(\Delta\rho) = -0.0450\, x_t$. For comparative purposes, we rescale this result to the case $\alpha_s(m_Z) = 0.118$, the value used in our calculations, and obtain, for $m_t = \sqrt{3} m_Z$,

$$\delta(\Delta\rho)_{thr} = -0.0486\, x_t \qquad \text{(Ref. [6])}. \tag{8}$$

In our own work we have applied two different methods to evaluate the imaginary parts near threshold. The first one is the resonance approach of Ref. [4], which assumes the existence of narrow, discrete $t\bar{t}$ bound states characterized by $R_n(0)$ and M_n. Moreover, in

Ref. [4] a specific interpolation procedure is developed to implement the matching of the higher resonances and the continuum evaluated perturbatively to $\mathcal{O}(\alpha_s)$. The second one is the Green-function (G.F.) approach of Ref. [3], which takes into account the smearing of the resonances by the weak decay of its constituents and leads to a continuous excitation curve. Both approaches make use of realistic QCD potentials, the Richardson and the Igi-Ono potentials, which reproduce accurately charmonium and bottonium spectroscopy and are expected to describe well toponia, too. These QCD potentials contain a term linear in the inter-quark distance, r, to account for the confinement of color. Detailed studies reveal that the shape of the Green function is not very sensitive to the long-distance behaviour of the potential [12]. This may be understood by observing that the top quarks decay before they are able to reach large distances. The rapid weak decay of the top quarks, which causes the screening of the long-distance effects, is properly taken into account in the Green-function approach of Ref. [5], while it is not implemented in the resonance approaches of Refs. [5] and [6]. In the latter case, it is clearly more consistent theoretically and more realistic phenomenologically to keep the linear term of the potential, as is done in Ref. [5] but not in Ref. [6]. On the other hand, both resonance and Green-function approaches of Ref. [5] effectively resum the contributions of soft multi-gluon exchanges in the ladder approximation [3,13]. This automatically includes the final-state interactions emphasized in Ref. [6]. However, we stress that all these methods are based on a non-relativistic approximation. There are additional contributions due to the exchange of hard gluons, which give rise to sizeable reduction factors [14], e.g., $(1 - 3C_F\alpha_s/\pi)$ in the case of $\Delta\rho$ [5].

Because of the more complicated potentials used in our two approaches, we have to rely on numerical computations. For $m_t = \sqrt{3}m_Z$ and $\alpha_s(m_Z) = 0.118$, we find

$$\begin{aligned} \delta(\Delta\rho)_{thr} &= -(0.034 \pm 0.010)x_t \qquad \text{(G.F.)}, & (9)\\ \delta(\Delta\rho)_{thr} &= -(0.042 \pm 0.013)x_t \qquad \text{(res.)}, & (10) \end{aligned}$$

where we have included the 30% error estimate given in Ref. [5]. The corresponding perturbative $\mathcal{O}(\alpha_s)$ contribution [15] is, for $m_t = \sqrt{3}m_Z$,

$$\begin{aligned} \delta(\Delta\rho)_{\alpha_s} &= -\frac{2\alpha_s(m_t)}{3\pi}\left(\frac{\pi^2}{3} + 1\right)x_t \\ &= -0.0991\, x_t. \qquad (11) \end{aligned}$$

Unlike $\delta(\Delta\rho)_{thr}$, $\delta(\Delta\rho)_{\alpha_s}$ obtains important contributions arising from the non-conserved vector and axial-vector currents associated with the W-boson vacuum-polarization function.

It is apparent that the result for $\delta(\Delta\rho)_{thr}$ obtained in the formulation of Ref. [6] (Eq. (8)) is of the same magnitude as our two evaluations (Eqs. (9), (10)). It amounts to 49% of the $\delta(\Delta\rho)_{\alpha_s}$ correction, while our results of Eqs. (9), (10) correspond to 34% and 42%, respectively. Thus, it is somewhat larger than our resonance calculation and significantly larger than our G.F. result. Part of the difference is due to the fact that we have included a hard-gluon correction [14], $(1 - 3C_F\alpha_s(M_n)/\pi)$ (cf. Eqs. (4.2b, 4.3d) of Ref. [5]), which has not been incorporated into Eqs. (3), (4). Note that Eq. (5) must not be multiplied by this factor, since only non-relativistic terms are subtracted in the

TABLE I. $t\bar{t}$ threshold effects on $\Delta\rho$ relative to the Veltman correction, $-100 \times \delta(\Delta\rho)_{thr}/x_t$, as a function of m_t. The calculation based on Ref. [6] (total [6]) is compared with our previous resonance (res. [5]) and G.F. (G.F. [5]) results. For completeness, the contributions from below (pole [6]) and above (thr. [6]) threshold are also displayed separately in the first case. The hard-gluon correction is included in the three calculations and the input value $\alpha_s(m_Z) = 0.118$ is used.

m_t [GeV]	pole [6]	thr. [6]	total [6]	res. [5]	G.F. [5]
120.0	2.84	1.99	4.83	4.81	3.58
140.0	2.73	1.92	4.65	4.43	3.43
157.9	2.64	1.86	4.50	4.17	3.36
180.0	2.55	1.79	4.35	3.91	3.35
200.0	2.47	1.75	4.22	3.71	3.41
220.0	2.41	1.70	4.11	3.53	3.51

TABLE II. Perturbative $\mathcal{O}(\alpha_s)$ and $t\bar{t}$ threshold [6] contributions to $\Delta\rho$ relative to the Veltman correction, $-100 \times \delta(\Delta\rho)_{\alpha_s}/x_t$ and $-100 \times \delta(\Delta\rho)_{thr}/x_t$, for $m_t = \sqrt{3}m_Z$ as a function of μ_{pert}. The input value $\alpha_s(m_Z) = 0.118$ is used.

μ_{pert}	pert. $\mathcal{O}(\alpha_s)$	total [6]	sum
$m_t/2$	10.96	4.05	15.00
m_t	9.91	4.50	14.42
$2m_t$	8.87	5.18	14.05

integrand of that equation. If this correction is applied to Eqs. (3), (4), $\delta(\Delta\rho)_{pole}$ becomes $-0.0245\, x_t$ and, rescaled to $\alpha_s(m_Z) = 0.118$, the overall result in the formulation of Ref. [6] is $-0.0450\, x_t$, instead of Eq. (8). This is quite close to our resonance calculation (Eq. 10). For the reasons explained in Ref. [5] (see also the discussion), for values of $m_t \geq 130$ GeV we have expressed a preference for the G.F. approach. On the other hand, the three calculations amount to only 3.4%–4.5% of the $\mathcal{O}(\alpha)$ contribution, x_t.

The above results have been obtained for $m_t = \sqrt{3}m_Z$, the value employed in Ref. [6]. In Table I, we compare the calculation of $\Delta\rho$ using the formulation of Ref. [6] with our own resonance and G.F. evaluations, over the range 120 GeV $\leq m_t \leq$ 220 GeV. We have checked that the numbers given in the third and fourth columns of Table I do not change when we identify v_0 in Eq. (5) with the zero of the integrand, i.e., the point where the resummation of $(\pi C_F \alpha_s/v)^n$ terms matches the perturbative expression. It is apparent from Table I that the general features described above hold over the large range 120 GeV $\leq m_t \leq$ 220 GeV. In fact, the three calculations are similar in magnitude. Moreover, the formulation of Ref. [6] gives results somewhat larger but quite close to our resonance calculation, the agreement being particularly good at low m_t values, where the resonance picture is expected to work best.

In order to illustrate the stability of the results with respect to a change of the scale, μ_{pert}, employed in the perturbative $\mathcal{O}(\alpha_s)$ calculations, in Table II we show the values of $\delta(\Delta\rho)_{\alpha_s}$ (cf. Eq. (11)), $\delta(\Delta\rho)_{thr}$, and their sum for $m_t = \sqrt{3}m_Z$ and $\mu_{pert} = m_t/2, m_t, 2m_t$. Here $\delta(\Delta\rho)_{thr}$ is calculated on the basis of Eqs. (3) and (5), with $\alpha_s(m_t)$ replaced by $\alpha_s(\mu_{pert})$ in the last term of Eq. (5), and v_0 chosen as the zero of the integrand (the factor

$(1-v^2/3)/(1-v^2)^2$ discussed before Eq. (8) is also appended). We see that there are variations in $\delta(\Delta\rho)_{thr}$ of -10% to 15%, which are not particularly large. Interestingly, they partly compensate the corresponding variations in $\delta(\Delta\rho)_{\alpha_s}$. Indeed, the overall QCD correction, $\delta(\Delta\rho)_{\alpha_s}+\delta(\Delta\rho)_{thr}$, which is the physically relevant quantity, changes by only -3% to 4%, a remarkably small variation.

3 Bohr-atom estimate and other observations

It is instructive to make a simple estimate of the threshold effects on $\Delta\rho$ by using an elementary Bohr-atom model of toponium [1]. We have already employed this model to show that it leads to values of $|R_{10}(0)|^2$ within 20% of those obtained with the Richardson potential [16]. Now we want to apply it to illustrate the order of magnitude of $\delta(\Delta\rho)_{thr}$. Then, instead of Eq. (3), we have

$$\delta(\Delta\rho)_{thr} = -x_t\,\pi C_F^3 \sum_n \frac{\alpha_s^3(k_n)}{n^3}, \tag{12}$$

where $k_n = C_F\alpha_s(k_n)m_t/(2n)$ is the momentum of the top quark in the nS orbital of the Bohr-atom model. We note that in this elementary estimate we have evaluated α_s at scale k_n. This is a simple generalization of Ref. [1], where, for the ground state, α_s is evaluated at $k_1 = C_F\alpha_s(k_1)m_t/2$. In particular, for $m_t=\sqrt{3}m_Z$ and $\alpha_s(m_Z)=0.118$, we have $k_1 = 16.7$ GeV and $\alpha_s(k_1)=0.159$. In the rough estimate of Eq. (12), we have also disregarded the continuum enhancement above threshold. Because k_n scales as $1/n$, $\alpha_s(k_n)$ increases with n. We have iteratively evaluated k_n and $\alpha_s(k_n)$ up to $n=50$ and found $\sum_{n=1}^{50}\alpha_s^3(k_n)/n^3 = 5.56\times10^{-3}$, which is 1.38 times the $1S$ contribution. The sum converges rapidly; the first 12 terms already yield a factor of 1.36. Taking this to be an approximate estimate of the enhancement factor due to the resonances with $n\geq 2$, and normalizing Eq. (12) relative to Eq. (11), we have

$$\frac{\delta(\Delta\rho)_{thr}}{\delta(\Delta\rho)_{\alpha_s}} = 11.3\frac{\alpha_s^3(k_1)}{\alpha_s(m_t)}. \tag{13}$$

For $m_t=\sqrt{3}m_Z$ and $\alpha_s(m_Z)=0.118$, we have $\alpha_s(m_t)=0.109$, $\alpha_s(k_1)=0.159$, and Eq. (13) gives 42%, which is rather close to the results obtained from the detailed resonance approaches, i.e., Eqs. (8) ,(10) divided by Eq. (11). Although Eq. (13) is a rough estimate, it allows us to understand why the threshold effects on the ρ parameter, although nominally of $\mathcal{O}(\alpha_s^3)$, can be as large as $\approx 40\%$ of the $\mathcal{O}(\alpha_s)$ contribution. Two factors are apparent: one is a large numerical coefficient, ≈ 11, and the other is that the natural scale in the threshold contribution is $k_1 \ll m_t$, so that $\alpha_s(k_1)$ is considerably larger than $\alpha_s(m_t)$.

A relevant question is how these effects compare with unknown $\mathcal{O}(\alpha_s^2)$ contributions. The leading $\mathcal{O}(\alpha_s)$ corrections to the ρ parameter are $\approx 10\%$ (cf. Eq. (11)). If the same ratio holds between $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s)$, the threshold corrections we have discussed would be roughly 3 to 4 times larger for $m_t\approx 160$ GeV. It is known that the use of the running top-quark mass absorbs most of the $\mathcal{O}(\alpha_s)$ corrections proportional to m_t^2 in the ρ

parameter and $Z \to b\bar{b}$ amplitudes. If this was a general feature of the perturbative expansion, one could control the bulk of such contributions and the threshold effects would neatly stand out. However, it is impossible to ascertain these features without detailed $\mathcal{O}(\alpha_s^2)$ calculations, which are not available at present. The effect of these threshold corrections on the electroweak parameters are not particularly large for $m_t \leq 220$ GeV. For example, for $m_H = 250$ GeV and $m_t = (130, 160, 200)$ GeV we found [16] shifts $\Delta m_W = -(42, 55, 77)$ MeV from the $\mathcal{O}(\alpha_s)$ contributions, $\Delta m_W = -(14, 19, 27)$ MeV from threshold effects evaluated with the resonance method, and $\Delta m_W = -(10, 16, 25)$ MeV from threshold effects evaluated in the G.F. approach. The ratio of threshold to $\mathcal{O}(\alpha_s)$ shifts in m_W are similar but not identical to those we encountered in $\Delta\rho$. The reason is that the range of m_t values of interest is not really in the asymptotic regime, and the subleading $\mathcal{O}(\alpha_s)$ and threshold contributions to Δr [17] (the relevant correction to calculate m_W) have somewhat different m_t dependences. For example, for $m_t = 160$ GeV and $m_H = 250$ GeV, the ratios for m_W shifts are 0.35 in the resonance approach and 0.29 in the G.F. method [16]. For $m_t = m_Z$ (220 GeV), the corresponding ratios amount to 0.25 (0.34) in the resonance approach and to 0.19 (0.34) in the G.F. method, with the latter giving somewhat lower values below $m_t = 220$ GeV.

4 Conclusions and discussion

1. We have pointed out that the conclusions of Ref. [6] concerning the magnitude of the threshold effects discussed in Ref. [5] are without foundation. They are based on the consideration of a vector amplitude $\Pi(s) - \Pi(0)$, which, although very important in other applications, has a very small effect in the analysis of Ref. [5]. This should be obvious because heavy particles of mass $m^2 \gg s$ decouple in this amplitude.

2. We have applied the analysis of $\Im m\, \Pi_{thr}(s')$ given in Ref. [6] to study the threshold corrections to the ρ parameter for $m_t = \sqrt{3} m_Z$ and, after re-evaluating the quantities involved, found a result that is similar in magnitude to those we have reported previously. Contrary to the conclusions of Ref. [6], it is somewhat larger than our resonance calculation and significantly larger than our Green-function (G.F.) evaluation. In particular, when hard-gluon contributions are included, the result derived in the approach of Ref. [6] is quite close to our own resonance calculation, well within the theoretical errors quoted previously [5]. As illustrated in Table I, similar conclusions apply over the large range 120 GeV $\leq m_t \leq$ 220 GeV. We have also pointed out that the formulation of Ref. [6] leads, for $m_t = \sqrt{3} m_Z$, to values of $\delta(\Delta\rho)_{\alpha_s} + \delta(\Delta\rho)_{thr}$ which are remarkably stable with respect to changes in the scale employed in the perturbative calculation (see Table II).

3. In our opinion, the fact that the formulation of Ref. [6], when applied to $\Delta\rho$, gives results similar to ours (and, in fact, quite close to our own resonance calculations), supports the notion that these threshold effects can be reasonably estimated. By the same token, it does not support recent claims of large ambiguities in the threshold calculations [18].

4. It is worthwhile to point out that, when the corrected values of δ_{pole} and δ_{thr} obtained in the present paper are applied, the threshold effects in $\Pi(s) - \Pi(0)$ amount to $\approx$ 24% of the perturbative $\mathcal{O}(\alpha_s)$ corrections evaluated at scale m_t. Here $m_t = \sqrt{3} m_Z$ and $\alpha_s(m_Z) = 0.115$ have been assumed and, for simplicity, the hard-gluon correction is not included. (The numerical evaluation reported in Ref. [6] gives, instead, 15%). These numbers are somewhat smaller than the effects we encountered in the $\Delta\rho$ case: 34% in the G.F. approach and 42% in the resonance framework (cf. Eqs. (8), (10) divided by Eq. (11)). However, we find it neither extraordinary nor unusual that radiative-correction effects may vary by factors of 1.42, 1.75 or, for that matter, 2.8, when applied to very different amplitudes. In particular, the logic behind the conclusion of Ref. [6] seems rather strange to us. It is apparently based on the curious argument that a 15% correction in $\Pi(s) - \Pi(0)$ is considered to be very small and that, as a consequence, a 34% or 42% effect in $\Delta\rho$ is intolerably large.

5. It is well known that, for large m_t, the widths of the individual top quarks become larger than the $1S$–$2S$ mass difference, so that the bound-state resonances lose their separate identities and smear into a broad threshold enhancement [3]. For this reason, we have expressed a preference to use the resonance formulation for $m_t \leq 130$ GeV and the G.F. approach for $m_t \geq 130$ GeV [5]. However, we do not regard either method as "rigorous" and, in fact, in Ref. [5] we have assigned an estimated 30% uncertainty to their evaluation. In the analysis of electroweak parameters, we have found that the resonance approach of Ref. [4] and the G.F. formulation of Ref. [3] lead to similar results over a wide m_t range, with the latter giving somewhat smaller values for $m_t \leq 220$ GeV.

6. In order to make the relative size of the threshold effects more readily understandable, we have presented a simple estimate based on an elementary Bohr-atom model of toponium (cf. Eq. (13)), and briefly commented on the possible magnitude of $\mathcal{O}(\alpha_s^2)$ corrections.

7. We have not discussed here the non-perturbative contributions connected with the existence of a gluon condensate, since they are known to be exceedingly small in the case of the $t\bar{t}$ threshold [19]; this has also been noticed in Ref. [6].

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[7] In Ref. [6] there is a footnote claiming that an expression in Ref. [5] involving $|R_n(0)|^2$ is not renormalization-group invariant. This is not the case, as in our discussion $|R_n(0)|^2$ and the other terms in that expression are evaluated at their physical scales (i.e., they do not depend on an arbitrary scale μ). The same would occur in Eq. (2) if μ was identified with the relevant physical scales, which are roughly the top-quark momenta in the Bohr-atom orbitals. In Eq. (2), the explicit μ dependence compensates for the fact that $|\tilde{R}_{n0}^{(0)}|^2$ is evaluated at scale μ.

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[10] Had we used an arbitrary scale μ in the calculation of the last term in Eq. (5), this requirement would no longer be satisfied.

[11] In Ref. [6], the numerical value $\delta_{thr} = 5.4 \times 10^{-3}$ is reported, which is neither consistent with Eq. (5) nor with Eq. (7).

[12] See, e.g., Fig. 33 in W. Bernreuther et al., *Proceedings of the Workshop on e^+e^- Collisions at 500 GeV: The Physics Potential*, Munich, Annecy, Hamburg, 1991, ed. by P.M. Zerwas, DESY Report No. 92–123A (August 1992) p. 255.

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